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On the 9-total-colorability of planar graphs with maximum degree 8 and without intersecting triangles

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ABSTRACT

Recently, Kowalik, Sereni, and Škrekovski proved that planar graphs with maximum degree 9 are 10-totally colorable. This work proves that planar graphs with maximum degree 8 and without intersecting triangles are 9-totally colorable.

1. Introduction

All graphs considered in this work are finite, simple and undirected. Undefined terminology and notation in this work can be found in the book by Bondy and Murty [1].

A graph is *planar* if it can be embedded into the plane so that its edges only meet at their ends. Any such concrete embedding of a planar graph is called a *plane* graph. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, F, Δ and δ), respectively. A k-cycle is a cycle of length k. A 3-cycle is also called a triangle. Two triangles are *intersecting* if they have at least one vertex in common.

A *k-total-coloring* of a graph G is a mapping ϕ from $V \cup E$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every pair of adjacent or incident elements $x, y \in V \cup E$. The graph G is called *k-totally colorable* if it has a k-total-coloring. It is clear that at least $\Delta + 1$ colors are needed to color G totally. Vizing [2] and Behazd [3] independently conjectured that every graph is $(\Delta + 2)$-totally colorable. This conjecture is known as the Total Coloring Conjecture (TCC).

TCC was extensively researched in the literature, see [4–9]. Even for planar graphs, TCC remains open, see [10–13,9]. So far, we have not found any planar graph with maximum degree $\Delta \geq 4$ which is not $(\Delta + 1)$-totally colorable. On the total colorability of planar graphs, one often hopes to get the best result, namely, to prove that the planar graphs under consideration are $(\Delta + 1)$-totally colorable. Borodin et al. [14], Wang [16] and Kowalik et al. [15] successively proved that planar graphs with $\Delta \geq 11$, $\Delta = 10$ and $\Delta = 9$ are $(\Delta + 1)$-totally colorable. As far as we know, it is unknown whether all planar graphs with maximum degree 8 are 9-totally colorable! This work shows a moderate result on the topic as follows:

Theorem 1. *Every planar graph G with $\Delta = 8$ and without intersecting triangles is* 9*-totally colorable.*

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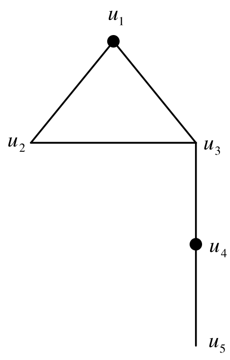

Fig. 1. Forbidden configurations in G.

2. Reducible configurations

Let G be a counterexample to Theorem 1 with $\sigma(G) = |V| + |E|$ as small as possible. That is, any proper subgraph of G is 9-totally colorable while G itself is not. Embed G into the plane. It is easy to see that G is 2-connected. Hence, G has no vertices of degree 1 and the boundary of every face of G is a cycle.

For a face $f \in F$, its degree, denoted by $d(f)$, is defined to be the length of its boundary walk. Call f a *k-face*, a *k^+-face*, or a *k^--face* if $d(f) = k$, $d(f) \geq k$, or $d(f) \leq k$, respectively. The concepts of a *k-vertex*, a *k^+-vertex*, and a *k^--vertex* are similarly defined.

Lemma 1. (a) *Let $uv \in E(G)$. If $d(u) \leq 4$, then $d(u) + d(v) \geq 10$.*
(b) *The subgraph induced by all edges joining 2-vertices to Δ-vertices in G is a forest.*

The proof of Lemma 1 can be found in [14]. As corollaries of Lemma 1(a), the two neighbors of a 2-vertex are 8-vertices; the neighbors of a 3-vertex are 7^+-vertices; and the neighbors of a 4-vertex are 6^+-vertices in G.

Let H be the forest stated in Lemma 1(b) and T, a maximal tree in H. It is easy to see that all leaves of T are 8-vertices. It is easy to prove by induction that T has a maximum matching, say, M, that matches every 2-vertex of T and all 8-vertices but one that is M-unmatched. The unmatched 8-vertex by M is called the root of T. Fix a maximum matching for each maximal tree in H. Let v be a 2-vertex in G. The neighbor of v which is matched to v by the prescribed matching is called the *master* of v. The concept of the master was first introduced in [14] and will play an important role in the next section.

Lemma 2. *G contains no configurations depicted in Fig. 1 where the vertices marked by • have no other neighbors in G.*

Proof. Suppose that G has the configurations depicted in Fig. 1. By the minimality of $\sigma(G)$, $G' = G - u_1u_3$ has a 9-total-coloring $\phi : V(G') \cup E(G') \longrightarrow \{1, 2, \ldots, 9\}$. Erase the colors of u_1 and u_4. Let $S(u)$ be the set of colors used by ϕ for edges incident with u, and $\overline{S}(u) = S(u) \cup \{\phi(u)\}$. If $\phi(u_1u_2) \in \overline{S}(u_3)$, then there are at most eight forbidden colors for u_1u_3, i.e., u_1u_3 can be properly colored. Clearly, u_1 and u_4 can be properly recolored easily. Hence, G is 9-totally colorable, a contradiction. So, we can assume that $\phi(u_1u_2) \notin \overline{S}(u_3)$. Without loss of generality, assume $\overline{S}(u_3) = \{1, 2, \ldots, 8\}$, $\phi(u_1u_2) = 9$. If $\phi(u_4u_5) \neq 9$, color u_1u_3 with $\phi(u_3u_4)$ and recolor u_3u_4 with 9, then G is 9-totally colorable, a contradiction. Assume $\phi(u_4u_5) = 9$. At this time, we first recolor edges u_1u_2, u_2u_3, u_3u_4 with $\phi(u_2u_3)$, 9, $\phi(u_2u_3)$, respectively, then color u_1u_3 with $\phi(u_3u_4)$. It follows that G is 9-totally colorable, a contradiction showing that G has no configuration in Fig. 1. □

Call a 3-face *bad* if it has a 3^--vertex. Similarly, a 4-face is *bad* if it has two 3^--vertices (note that they are non-adjacent by Lemma 1).

Lemma 3. *Let v be an 8-vertex incident with a bad 3-face T. If v is adjacent to a 2-vertex not on T, then v is incident with at most five bad 4-faces.*

Proof. Note that if v is incident with more than five bad 4-faces, then v is in one of the six configurations depicted in Fig. 2. Suppose that G has the configuration shown in Fig. 2(1). By the minimality of $\sigma(G)$, $G' = G - vv_1$ has a 9-total-coloring $\phi : V(G') \cup E(G') \longrightarrow \{1, 2, \ldots, 9\}$. Erase the colors of all the adjacent 3^--vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and v_8.

If $\phi(v_1u_1) \in \overline{S}(v)$, then the forbidden colors for vv_1 are at most 8, i.e., vv_1 can be properly colored. After recoloring the erased vertices, we obtain a 9-total-coloring of G, a contradiction. If $\phi(v_1u_1) \notin \overline{S}(v)$, we can assume $\overline{S}(v) = \{1, 2, \ldots, 8\}$ with $\phi(vv_i) = i - 1$, $i = 2, 3, \ldots, 8$, $\phi(v) = 8$ and $\phi(v_1u_1) = 9$. Note that $\phi(u_1v_8) \neq 9$. If $\phi(v_7v_8) \neq 9$, color vv_1 with $7 = \phi(vv_8)$ and recolor vv_8 with 9, getting a 9-total-coloring of G, a contradiction. So, we can assume $\phi(v_7v_8) = 9$. At this time, $\phi(v_7) \neq 9$. We first recolor v with 9 and then color vv_1 with 8, easily getting a 9-total-coloring of G, a contradiction showing that G has no configuration in Fig. 2(1).

All the configurations shown in Fig. 2(2)–(6) can be similarly proved. We omit the details here. □

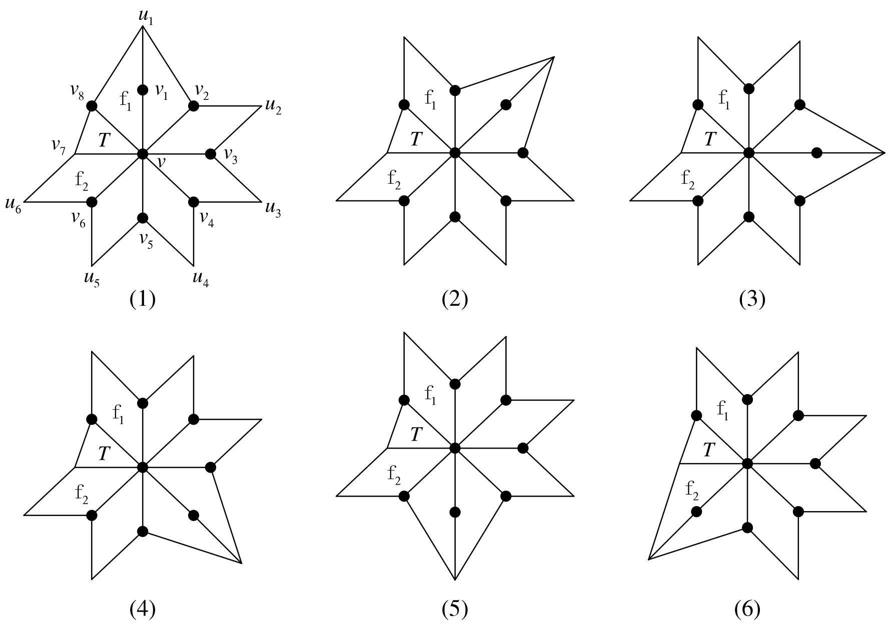

Fig. 2. v is incident with a bad 3-face and six bad 4-faces.

3. Discharging

To complete the proof of Theorem 1, we shall derive a contradiction by a discharging procedure based on the structural properties of G that were established in Section 2. The *initial charge function ch* in the discharging procedure is defined as

$$ch(x) = \begin{cases} 2d(x) - 6, & \text{if } x \in V; \\ d(x) - 6, & \text{if } x \in F. \end{cases}$$

By Handshaking Lemmas $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} d(f)$ and Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{x \in V \cup F} ch(x) = \sum_{x \in V} (2d(x) - 6) + \sum_{x \in F} (d(x) - 6) = -12.$$

Since any discharging procedure preserves the total charge of G, if we can define suitable discharging rules to change the initial charge function to the *final charge function* ch' on $V \cup F$ such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$, which completes our proof.

Now, we define the discharging rules as follows:

R1: *Charge to a 2-vertex* v.
R1.1. If v is on a 3-face, then it gets 1 from each of its incident vertices.
R1.2. If v is not on a 3-face, then it gets 2 from its master.
R2: *Charge to a 3-face*.
R2.1. A 3-face incident with a 4^--vertex gets $\frac{3}{2}$ from each of its two incident 6^+-vertices.
R2.2. A 3-face not incident with any 4^--vertex gets 1 from each of its incident vertices.
R3: *Charge to a 4-face*.
R3.1. A bad 4-face gets 1 from each of its two incident 7^+-vertices.
R3.2. A 4-face incident with only one 3^--vertex gets $\frac{3}{4}$ from each of its two incident 7^+-vertices; $\frac{1}{2}$, from the left incident 4^+-vertex.
R3.3. A 4-face not incident with any 3^--vertex gets $\frac{1}{2}$ from each of its incident vertices.
R4: *Charge to a 5-face*.
R4.1. A 5-face incident with two 3^--vertices gets $\frac{1}{3}$ from each of its incident 7^+-vertices.
R4.2. A 5-face incident with only one 3^--vertex gets $\frac{1}{4}$ from each of its incident 4^+-vertices.
R4.3. A 5-face not incident with any 3^--vertex gets $\frac{1}{5}$ from each of its incident vertices.

These rules are illustrated in Fig. 3.

The rest of this work is devoted to checking that $ch'(x) \geq 0$ for all $x \in V \cup F$. First note that our discharging rules are just designed so that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices in V. So we only need to check that $ch'(v) \geq 0$ for all 3^+-vertices in G.

Remark 1. $\forall v \in V$, there is at most one 3-face incident with v for G having no intersecting triangles.

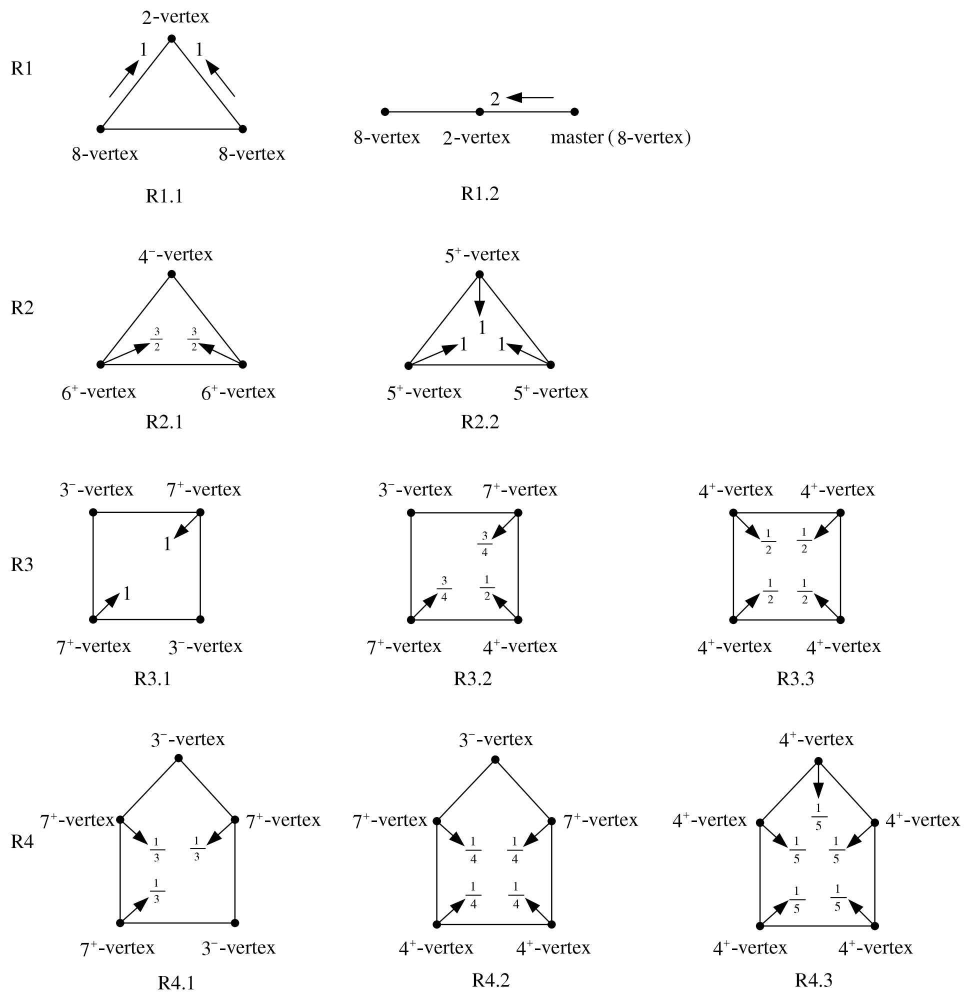

Fig. 3. Discharging rules.

Remark 2. A 7^+-vertex discharges nothing to each of its incident 6^+-faces; at most 1 to each of its incident bad 4-faces; at most $\frac{3}{4}$ to each of its incident non-bad 4-faces.

Let $d(v) = 3$. According to our discharging rules, no charge is discharged to or from v, that is, $ch'(v) = ch(v) = 0$.

Let $d(v) = 4$. By R2.1, R3.2, R3.3, R4.2 and R4.3, v discharges at most $\frac{1}{2}$ to each of its incident faces. So, $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$.

Let $d(v) = 5$. By our rules, v discharges at most 1 to a possible incident 3-face, and at most $\frac{1}{2}$ to each of the left 4^+-faces. Thus, $ch'(v) \geq ch(v) - 1 - \frac{1}{2} \times 4 = 1$.

Let $d(v) = 6$. By our rules, v discharges at most $\frac{3}{2}$ to a possible incident 3-face, and at most $\frac{1}{2}$ to each of the left 4^+-faces. It follows that $ch'(v) \geq ch(v) - \frac{3}{2} - \frac{1}{2} \times 5 = 2$.

Let $d(v) = 7$. By Remarks 1 and 2, $ch'(v) \geq ch(v) - \frac{3}{2} - 1 \times 6 = \frac{1}{2}$.

Let $d(v) = 8$. If v is not adjacent to any 2-vertex, then $ch'(v) \geq ch(v) - \frac{3}{2} - 1 \times 7 = \frac{3}{2}$. If v is not incident with any 3-faces, by R1 and Remark 2, $ch'(v) \geq ch(v) - 2 - 1 \times 8 = 0$. Assume that v is incident with a 3-face, say, T, and adjacent to at least one 2-vertex. If T has a 2-vertex, say, u, then, by Lemma 2, v is not incident with any other 2-vertex, namely, the maximal tree containing u in H is a path of length 2. According to R1.1, $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - 1 \times 7 = \frac{1}{2}$.

Now, we can assume that T has no 2-vertex. If T is not a bad 3-face, then both the two faces adjacent to T and incident with v, say, f_1 and f_2, are not bad 4-faces. By our rules and, in particular, by Remark 2, v discharges at most $\frac{3}{4}$ to each of f_1 and f_2. Thus, $ch'(v) \geq ch(v) - 2 - \frac{3}{2} - 1 \times 5 - \frac{3}{4} \times 2 = 0$. If $T = vxy$ is a bad 3-face with a 3-vertex, say, x, by Lemma 3, all the faces incident with v other than T are 4^+-faces and the number of the bad 4-faces is at most 5. By Remarks 1 and 2, $ch'(v) \geq ch(v) - 2 - \frac{3}{2} - 1 \times 5 - \frac{3}{4} \times 2 = 0$. This concludes the proof of the theorem. □

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